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On thermally forced stratified rotating fluids

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Axisymmetric steady motion of an inhomogeneous rotating fluid is considered. A system of equations, with appropriate boundary conditions, controlling the smooth interior fields is derived under the assumption of small dissipation and small side-boundary conductance. It is argued that this system, being derived without linearization of the equations, might form the basis of valuable numerical analysis. Assuming sufficiently weak forcing, i.e. high insulation of the non-horizontal boundaries, a linear system is derived. An explicit solution is presented and discussed for a particularly simple and important case.

1. Introduction

We consider in this paper axisymmetric steady motion of an inhomogeneous rotating fluid. We deal only with thermally forced flow.

Non-axisymmetric motion has been the object for extensive though mostly experimental work in the past (see, e.g., Fowlis & Hide 1965). The theoretical basis for these experiments has been rather weak, in fact not even the basic symmetric state has been fully understood. Previous thoeretical studies of steady flow include some papers by Barcilon & Pedlosky (1967*a*, *b*). Their analysis differs most importantly from the present one in that they linearize the differential equations from the outset. This procedure leads to an extremely restrictive condition on the magnitude of the fluid motion that can be allowed.

McIntyre (1968) analysed a much more realistic flow régime. His analysis, although physically most clarifying, was not entirely complete from a theoretical point of view (and could not have been because of certain basic difficulties *op cit.*, p. 644). It is different from the present analysis in that unlike McIntyre, we have introduced the finite conductance of the bounding surfaces as a parameter to control the rate of thermal forcing, which leads to a completely tractable theory. In this respect the present paper is a natural continuation of the analysis given by Walin (1971), where similar but non-rotating systems were considered.

In §2 the basic equations and boundary conditions are presented: we adopt the Boussinesq approximation. The equations are non-dimensionalized and the basic parameters presented. Among these is the Rossby number, R, based on an unknown velocity scale. It is pointed out that R is not really independent but is essentially controlled by properties of the boundary, i.e. the parameter sL which depends on the conductance of the boundary. The quantity s^{-1}, as defined by equation (2.2*c*) may be thought of as the 'thermal thickness' of the boundary while L is a characteristic length scale of the container.

In §3 we explore the consequences of letting the dissipation parameter, E, become very small. It is found that the system degenerates to a state characterized by a possibly smooth interior with thin boundary layers in the vicinity of the boundaries. The equations governing the interior fields (ϕ^I) are found to become 'semi-degenerate' in that two of the original four dissipative terms disappear, namely those in the radial and vertical momentum equations (2.1 *a*, *c*). In general, the interior equations remain nonlinear, although the remaining nonlinearity can be expected to be much less severe than the original one.

The boundary-layer equations are derived and it is found that without further assumptions they become linear to lowest order in E. In fact they differ little from the ordinary Ekman- and buoyancy-layer equations. In the derivation of the boundary-layer equations for small E we limit the analysis to a right circular cylinder.

It is found possible to integrate the boundary-layer equations separately. Consequently we are able to derive, in explicit form the boundary conditions to be satisfied by the degenerate set of differential equations controlling the interior fields, ϕ^I.

It is found that the original four boundary conditions (on the velocity vector and the temperature) reduce to two conditions on each boundary. The resulting system of equations for ϕ^I, being nonlinear, cannot be handled analytically in general. We think, however, that this system (or systems developed in a similar way) form a suitable basis for numerical analysis. The reason is that the greatest difficulty hampering any numerical description, namely the boundary-layer character of the solution, has been removed. This might perhaps open interesting possibilities for studying the nonlinear dynamics of the interior numerically.

In § 4 we discuss the relation between the Rossby number R and the properties of the boundary as expressed by sL. We then introduce an expansion in R into the system of equations and boundary conditions for ϕ^I derived in § 3. If the flow is characterized by $R \ll 1$ then the basic temperature field varies linearly or exponentially in the vertical and the horizontal temperature variations are small. Three cases are discussed separately; in the most important case we demonstrate explicitly how the linearization procedure leads to a well-posed boundary-value problem for the basic temperature field as well as the deviations therefrom, i.e. the baroclinic vortex motion.

We also discuss a rather peculiar case in which the boundary conditions are chosen to create a non-divergent vertical boundary layer. This discussion leads to an interesting qualitative prediction which could be relatively easy to verify experimentally.

In § 5 we discuss a very simple and straightforward example which is solved explicitly. The case chosen is simply a vertical cylinder stratified by a temperature difference between top and bottom but having imperfect sidewall insulation.

In § 6 we discuss some qualitative features illustrated by the example of § 5. Most importantly we find two separate vertical scales one being the width L of the cylinder, the other the Lineykin depth (Lineykin 1955) $L/(B\sigma)^{\frac{1}{2}}$, where B is a measure of the relative importance of stratification and rotation, and σ is the Prandtl number as defined by equations (2.6 *b*, *c*). In § 7 finally we discuss some appealing experimental possibilities and also the results of some preliminary experimental efforts of our own.

2. Basic equations

We consider steady, axisymmetric convection in a container rotating with angular velocity $\Omega\mathbf{k}$. We use a cylindrical co-ordinate system (r, ϑ, z). The rotation vector

$\Omega\mathbf{k}$, the gravitational acceleration $-g\mathbf{k}$ and the axis of symmetry are all assumed parallel to the z axis. We adopt the Boussinesq approximation and assume that the centrifugal acceleration is small compared with the gravitational acceleration. Under these circumstances the governing equations may be written

$$uu_r + wu_z - \frac{1}{r}v^2 - 2\Omega v = -\frac{1}{\rho_0}p_r + \nu\left(\nabla^2 - \frac{1}{r^2}\right)u, \tag{2.1a}$$

$$uv_r + wv_z + \frac{1}{r}uv + 2\Omega u = \nu\left(\nabla^2 - \frac{1}{r^2}\right)v, \tag{2.1b}$$

$$uw_r + ww_z - \frac{1}{\rho_0}g\alpha T = -\frac{1}{\rho_0}p_z + \nu\nabla^2 w, \tag{2.1c}$$

$$\frac{1}{r}(ur)_r + w_z = 0, \tag{2.1d}$$

$$uT_r + wT_z = \kappa\nabla^2 T, \tag{2.1e}$$

where (u, v, w) is the velocity vector in the co-ordinate system (r, ϑ, z); T and p are the deviations in temperature and pressure from the state of constant temperature T_0, density ρ_0, and pressure $p_0 = -\rho_0 gz + \text{constant}$; ν and κ are the diffusivities of momentum and heat while α is the coefficient of thermal expansion.

The fluid is bounded by a rigid semiconducting wall i.e. we have

$$\left.\begin{aligned}(u, v, w) &= 0, \\ \mathbf{n}.\nabla T &= s(T - \hat{T}),\end{aligned}\right\}\quad \text{on all boundaries,} \tag{2.2a, b}$$

where $\mathbf{n}$ is a unit vector normal to the boundary, pointing into the fluid region. $\hat{T}$ and s are prescribed functions of position on the boundary. Physically $\hat{T}$ may be thought of as the prescribed temperature outside the boundary while s is related to the thickness, d, of the boundary by the formula

$$s = \frac{\hat{k}}{k}\cdot\frac{1}{d}, \tag{2.2c}$$

where $\hat{k}$ and k are the heat-conductivities of the wall material and the fluid respectively. We introduce the following non-dimensional variables into equations (2.1) and (2.2).

$$(r, z) = L(r', z'), \tag{2.3a}$$

$$(u, v, w) = U(u', v', w'), \tag{2.3b}$$

$$p = Pp' = gL\alpha\Delta Tp', \tag{2.3c}$$

$$(T, \hat{T}) = \Delta T(T', \hat{T}'). \tag{2.3d}$$

We assume the aspect ratio to be of order one thus letting L represent the vertical as well as the horizontal scale. We anticipate that the strongest motion occurs in the zonal direction and define U to be scale of that motion i.e.

$$v' \sim 1, \tag{2.3e}$$

$$u', w' \lesssim 1. \tag{2.3f}$$

The scale P is chosen in anticipation of hydrostatic balance in the main part of the fluid region. Finally we assume that the temperature field and the prescribed function $\hat{T}$ are characterized by the same scale ΔT. Dropping the primes we thus obtain

$$R\left(uu_r + wu_z - \frac{v^2}{r}\right) - v = -\frac{B}{R}p_r + E\left(\nabla^2 - \frac{1}{r^2}\right)u, \tag{2.4a}$$

$$R\left(uv_r + wv_z + \frac{uv}{r}\right) + u = E\left(\nabla^2 - \frac{1}{r^2}\right)v, \tag{2.4b}$$

$$R(uw_r + ww_z) - \frac{B}{R}T = -\frac{B}{R}p_z + E\nabla^2 w, \tag{2.4c}$$

$$\frac{1}{r}(ur)_r + w_z = 0, \tag{2.4d}$$

$$R(uT_r + wT_z) = \frac{E}{\sigma}\nabla^2 T, \tag{2.4e}$$

and

$$\left.\begin{aligned} &(u, v, w) = 0, && (2.5a)\\ &\mathbf{n}.\nabla T = sL(T - \hat{T}) && (2.5b)\end{aligned}\right\} \text{ on all boundaries.}$$

Equations (2.4) contain the following externally controlled parameters

$$E = \nu/2\Omega L^2, \tag{2.6a}$$

$$B = g\alpha\Delta T/\rho_0 L(2\Omega)^2, \tag{2.6b}$$

$$\sigma = \nu/\kappa. \tag{2.6c}$$

The Rossby number

$$R = U/2\Omega L \tag{2.6d}$$

is not an independent parameter since the velocity scale is not externally controlled. In the boundary condition (2.5b) however we have the quantity sL which is a prescribed function of position on the boundary. The magnitude of sL (for which we are not introducing a separate symbol) serves as an external forcing parameter. A primary task for us is thus to find out how R (i.e. U) is related to sL.

3. The inviscid limit

3.1. *Interior equations*

We will next derive the equations governing our system in the inviscid limit. We thus assume

$$E \to 0 \tag{3.1a}$$

while the external quantities

$$(B\sigma \text{ and } sL) \text{ stay finite.} \tag{3.1b}$$

On R we impose the restriction

$$R \lesssim 1, \tag{3.1c}$$

a weak condition, which, however, in principle should be checked subsequently. Recalling that $v \sim 1$ by definition it is easily seen from (2.4) that

$$(u, w) \sim E. \tag{3.2a}$$

Making the substitution

$$(u, w, v, p, T) = (Eu^I, Ew^I, v^I, p^I, T^I) \tag{3.2b}$$

we find the following set of equations valid to lowest order in E.

$$-R\frac{(v^I)^2}{r} + v^I = \frac{B}{R}p_r^I, \tag{3.3a}$$

$$R\left(u^I v_r^I + w^I v_z^I + \frac{u^I v^I}{r}\right) + u^I = \left(\nabla^2 - \frac{1}{r^2}\right) v^I, \tag{3.3b}$$

$$T^I = p_z^I, \tag{3.3c}$$

$$\frac{1}{r}(u^I r)_r + w_z^I = 0, \tag{3.3d}$$

$$R\sigma(u^I T_r^I + w^I T^I) = \nabla^2 T^I. \tag{3.3e}$$

Equations (3.3) have lost two of the four dissipative terms present in (2.4). Thus the order of the system has been lowered and solutions to (3.3) are unable to satisfy the complete boundary conditions as given by (2.5). We thus expect boundary layers at least somewhere in the region. We note however that dissipation remains important in the interior irrespective of how small we make E. This phenomenon, typical for rotating non-homogeneous flow, has been pointed out earlier, see e.g. McIntyre (1968).

3.2. *The boundary layers – a right circular cylinder*

As already mentioned equations (3.3), although valid in the main part of the region, cannot describe the system everywhere.

Primarily we then expect the appearance of boundary layers in the vicinity of the rigid boundaries. We thus assume

$$\phi = \phi^I + \phi^B, \tag{3.4a}$$

where ϕ represents the complete solution for any dependent variable, ϕ^I represents a solution to equations (3.3) while ϕ^B is non-zero only in a thin layer close to the boundary. This definition is then to be interpreted quantitatively as

$$\nabla\phi^I \sim \phi^I, \quad \nabla\phi^B.\mathbf{n} \sim \delta^{-1}\phi^B, \quad \nabla\phi^B \times \mathbf{n} \sim \phi^B,$$

where $\delta \ll 1$ is the thickness of the boundary layer and where $\mathbf{n}$ is a unit vector normal to the boundary.

The equations governing ϕ^B is obtained in the following way:

(i) Insert $\phi = \phi^I + \phi^B$ into equations (2.4).

(ii) Subtract equations (3.3). Note that ϕ^I will remain in the resulting equations owing to the presence of nonlinear terms in (2.4).

(iii) Find the approximate form of these equations to lowest order, assuming E and δ to be small.

In this process a relation between E and δ is normally found. We thus obtain two sets of dependent variables, satisfying different sets of differential equations. These two sets of dependent variables should together satisfy the complete boundary conditions i.e. we have

$$\left.\begin{aligned}(Eu^I+u^B, Ew^I+w^B, v^I+v^B) &= 0 \\ \mathbf{n}.\nabla(T^I+T^B) &= sL(T^I+T^B-\hat{T})\end{aligned}\right\} \text{ on all boundaries} \quad \begin{aligned}(3.5a)\\(3.5b)\end{aligned}$$

Furthermore from the definition of ϕ^B we have an additional condition

$$(T^B, u^B, w^B, v^B) \to 0 \quad \zeta/\delta \to \infty \tag{3.5c}$$

where δ is the thickness of the boundary layer (in which ϕ^B is non-zero) and ζ is the distance from the boundary. Note that we can in general not solve the equation for ϕ^I separately since we have no separate boundary conditions for ϕ^I.

We now limit the analysis to a right circular cylinder

$$r \leqslant 1, \quad -h \leqslant z \leqslant h. \tag{3.6}$$

We thus have

$$(Eu^I+u^B,\ Ew^I+w^B,\ v^I+v^B) = 0 \quad \text{on} \quad \begin{cases} r = 1, & (3.7a) \\ z = \pm h, & (3.7b)\end{cases}$$

$$(T^I+T^B)_z = \mp (sL)_\pm (T^I+T^B-T_\pm) \quad \text{on} \quad z = \pm h, \tag{3.7c}$$

$$(T^I+T^B)_r = -(sL)_v (T^I+T^B-T_v) \quad \text{on} \quad r = 1, \tag{3.7d}$$

where we have introduced the notation

$$sL = (sL)_\pm \quad \text{on} \quad z = \pm h,$$
$$sL = (sL)_v \quad \text{on} \quad r = 1,$$
$$\hat{T} = T_\pm \quad \text{on} \quad z = \pm h,$$
$$\hat{T} = T_v \quad \text{on} \quad r = 1.$$

The region under consideration is illustrated in figure 1.

In the following we will assume that

$$(sL)_v \lesssim 1 \tag{3.8a}$$

in accordance with our general scaling requirement (3.1). We will however allow $(sL)_\pm$ to take on large values i.e.

$$(sL)_\pm \gtrsim 1 \tag{3.8b}$$

since this as we will find does not influence the procedure of approximation.

We now make a slight generalization in that we allow a volume flux M_0 to pass through the system. We thus assume that conditions (3.7) apply everywhere except at the corners where we allow fluid to enter or leave the system. Physically we imagine a narrow slot between the cylindrical bounday and the upper and lower end plates allowing fluid to enter and leave the system respectively. Of course, from continuity, if fluid enters at one corner an equal amount will leave at the other. This generalization follows naturally since we already have a discontinuity in our analytical description at

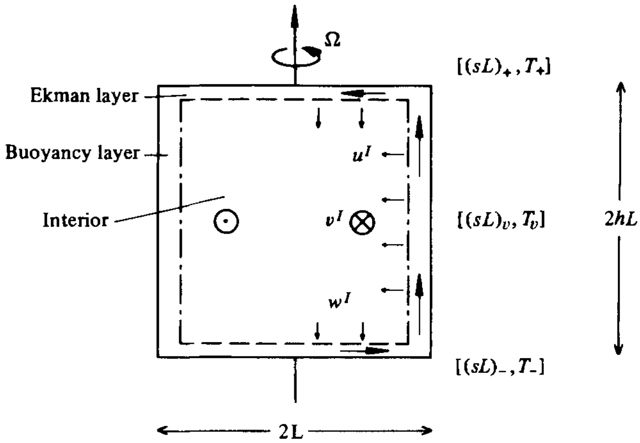

FIGURE 1. Illustration of the cylindrical region introduced in §3.2. $((sL)_\pm, T_\pm)$ and $((sL)_v, T_v)$ are the parameters entering the thermal boundary conditions (3.7*c*, *d*). The arrows indicate a possible flow pattern.

the corners where our different boundary-layer types meet. In fact the boundary-layer fluxes do not automatically match at the corner unless we make them do so with an extra continuity requirement. This means that we can without any complication prescribe a certain jump in the boundary-layer flux corresponding to a volume flux entering or leaving the system. Such a jump will automatically appear if we prescribe a net volume flow through the region.

To make our system complete we thus prescribe

$$R \int_0^1 (Ew^I + w^B)\, 2\pi r dr = M_0 E, \quad -h < z < h. \tag{3.9}$$

We assume that $M_0 \sim 1$, i.e. that the volume flux forced by M_0 is of the same order as that carried by w^I. Note that we have different scales for e.g. w^I and w^B. In fact for ϕ^B we will keep the scaling introduced in § 2 throughout the analysis to avoid extensive introduction of new notations.

3.3. *The vertical boundary-layer equations*

The procedure to derive the boundary-layer equations becomes relatively complicated because of the presence of the nonlinear terms in (2.4). The details are very similar to the case treated by Walin (1971) and will not be presented here. The result is however relatively simple. Thus we find:

$$p_r^B = 0, \quad v_{rr}^B = 0, \tag{3.10a,b}$$

$$-\frac{B}{R} T^B = E w_{rr}^B - \frac{B}{R} p_z^B, \tag{3.10c}$$

$$\frac{1}{r}(u^B r)_r + w_z^B = 0, \tag{3.10d}$$

$$R w^B T_z^I = \frac{E}{\sigma} T_{rr}^B. \tag{3.10e}$$

In order to keep down the number of symbols we have not introduced any special stretched boundary layer co-ordinate. This means that the small parameter E remains in the equations. The transformation required to eliminate E is $\eta = (1-r)E^{-\frac{1}{2}}$. We conclude that the boundary-layer thickness is given by

$$\delta = E^{\frac{1}{2}}. \tag{3.11}$$

Making use of condition (3.5c) i.e.

$$(p^B, v^B) \to 0 \quad \text{when} \quad (1-r)E^{-\frac{1}{2}} \to \infty,$$

equations (3.10) simplify to

$$p^B = 0, \quad v^B = 0, \tag{3.12a,b}$$

$$\frac{B}{R} T^B = E w^B_{rr}, \tag{3.12c}$$

$$\frac{1}{r}(u^B r)_r + w^B_z = 0, \tag{3.12d}$$

$$R w^B T^I_z = \frac{E}{\sigma} T^B_{rr}. \tag{3.12e}$$

We thus find that the vertical boundary layer is governed by the ordinary buoyancy-layer equations despite the nonlinearity of the interior dynamics. For the derivation of (3.10) it is necessary that

$$T^B \ll T^I. \tag{3.13}$$

This condition follows however from the boundary condition (3.7d) whenever

$$(sL)_v \ll \delta^{-1}$$

which in view of (3.11) is a consequence of (3.8). Note that equation (3.12a) should be interpreted in the sense that p^B is small enough to be neglected in (3.10c). If for some reason an evaluation of p^B is required we must of course bring in more terms in (3.10a). Note also that (3.12b) simply means that $v^B \ll 1$ i.e. $v^B \ll v^I$, a result of direct importance in the subsequent derivation of boundary conditions for ϕ^I.

3.4. *The horizontal boundary-layer equations*

Following a similar procedure as for the vertical boundary layer we obtain for the boundary-layer contribution in the vicinity of the horizontal boundaries:

$$-v^B = E u^B_{zz}, \quad u^B = E v^B_{zz}, \tag{3.14a, b}$$

$$p^B = 0, \tag{3.14c}$$

$$\frac{1}{r}(u^B r)_r + w^B_z = 0, \tag{3.14d}$$

$$R u^B T^I_r = \frac{E}{\sigma} T^B_{zz}. \tag{3.14e}$$

In the derivation of (3.14) it is necessary to require

$$(v^B, u^B) \ll 1 \tag{3.15a}$$

and

$$v^I \ll 1 \tag{3.15b}$$

within the boundary layer. Recalling that $v^I \sim 1$ at least in the main part of the interior it might appear as if (3.15) were hard to satisfy. This is not so however. In fact (3.15) is a consequence of previous assumptions and does not imply any further restriction on our theory. To see this it is sufficient to recall that

$$w^I \lesssim E$$

which, in accordance with well-known properties of the Ekman layer, implies that $(u^B, v^B, v^I) \sim E^{\frac{1}{2}}$ within the boundary layer. We thus have $v^I \sim 1$ in the interior and $v^I \sim E^{\frac{1}{2}}$ in the Ekman layer. Locally, in the vicinity of the boundary, v^I is thus much smaller than in the main part of the region. This is no contradiction but merely implies a homogeneous boundary condition on v^I as discussed in the next subsection.

Again we have not introduced a stretched boundary layer co-ordinate in (3.14). The transformation $\zeta = E^{-\frac{1}{2}}(h \mp z)$ eliminates E from (3.14). We conclude that we have the same boundary-layer thickness

$$\delta = E^{\frac{1}{2}} \tag{3.16}$$

for the horizontal as for the vertical boundary layer.

3.5. *Derivation of the boundary conditions for ϕ^I*

The main purpose of the boundary-layer analysis is to derive the correct boundary conditions for the interior part ϕ^I of the solution. Since equations (3.3) governing ϕ^I have a lower order than (2.4) the number of boundary conditions on ϕ^I must be smaller than on the complete solution $\phi = \phi^I + \phi^B$. In fact we will find that the original four conditions given by (3.7) are replaced by two conditions on each boundary to be applied on ϕ^I when solving the degenerate equations (3.3).

Let us begin with the vertical boundary. We thus derive an expression for the vertical boundary-layer flux in terms of ϕ^I and prescribed quantities. This may be obtained by integrating equation (3.12e) and by applying the thermal boundary condition. Recalling that $T^B \ll T^I$ we obtain

$$m^B = -\frac{E}{R\sigma}\left(\frac{(sL)_v\,(T^I - T_v) + T^I_r}{T^I_z}\right)_{r=1}, \tag{3.17a}$$

$$m^B = \int_{r_0}^{1} w^B dr \tag{3.17b}$$

where the integration is through the boundary layer, i.e.

$$(1 - r_0)\,E^{-\frac{1}{2}} \gg 1. \tag{3.17c}$$

From equation (3.12b) and the boundary condition on $v^I + v^B$ in (3.7a) we conclude that the appropriate boundary condition on v^I must be

$$v^I = 0 \quad \text{on} \quad r = 1 \tag{3.18a}$$

which in view of (3.3a, c) may be expressed in terms of T^I as

$$T^I_r = 0 \quad \text{on} \quad r = 1. \tag{3.18b}$$

The last term in (3.17a) is thus dropped and we obtain

$$m^B = -\frac{E}{R\sigma}\left(\frac{(sL)_v\,(T^I - T_v)}{T^I_z}\right)_{r=1}. \tag{3.19}$$

Applying the boundary condition on $u^I + u^B$ in (3.7a) and the continuity equation (3.12d) we obtain

$$R\sigma u^I = -\frac{d}{dz}\left(\frac{(sL)_v(T^I - T_v)}{T_z^I}\right) \quad \text{on} \quad r = 1. \tag{3.20}$$

We conclude that the boundary condition on $v^I + v^B$ simplifies to

$$v^I = 0 \quad \text{on} \quad z = \pm h. \tag{3.21}$$

Recalling also that $u^B \sim E^{\frac{1}{2}}$, $u^I \sim 1$ we conclude that the boundary condition on $Eu^I + u^B$ simplifies to

$$u^B = 0$$

while u^I is arbitrary on the horizontal boundary. The boundary condition on

$$w = Ew^I + w^B$$

gives rise to a relation between w^I and the boundary-layer flux. From continuity we obtain directly

$$2\pi r m^E = \pm \int_0^r Ew^I 2\pi r dr \quad \text{at} \quad z = \pm h \tag{3.22a}$$

$$m^E = \int_E u^B dz, \tag{3.22b}$$

where $\int_E$ denotes integration through anyone of the horizontal boundary layers.

Equation (3.22a) tells us how the boundary-layer flux m^E is determined by w^I. Integrating equations (3.14a, b, d) we may of course derive a relation between v^I and w^I familiar from the Ekman theory. The smallness of m^E as dictated by (3.22) would however only lead to the result $v^I = 0$, in consistency with (3.21). The boundary-layer flux m^E is thus determined by w^I rather than by v^I.

Finally we turn to the consequences of the thermal boundary condition. We thus integrate equation (3.14e) obtaining

$$RT_r^I m^E = \pm \frac{E}{\sigma} T_z^B \quad z = \pm h$$

which when combined with (3.22) and the thermal boundary condition (3.7c) yields

$$T_z^I + R\sigma T_r^I \frac{1}{r}\int_0^r w^I r\, dr = \mp (sL)_\pm (T^I - T_\pm) \quad \text{on} \quad z = \pm h. \tag{3.23}$$

Equations (3.21) and (3.23) are the appropriate boundary conditions to be satisfied by ϕ^I on the horizontal boundaries.

Finally we must express the 'overall continuity' condition (3.9) as a condition on ϕ^I. Making use of (3.19) we thus obtain from (3.9)

$$R\int_0^1 rw^I dr - \frac{1}{\sigma}\left(\frac{(sL)_v(T^I - T_v)}{T_z^I}\right)_{r=1} = \frac{M_0}{2\pi} \quad \text{when} \quad -h < z < h. \tag{3.24}$$

In summary our expectation is that ϕ^I may be determined from equations (3.3) together with conditions (3.18), (3.20), (3.21), (3.23) and (3.24), i.e. we expect this set of equations and boundary conditions to form a well-posed boundary-value problem.

4. Linearization of the inviscid limit

In this section we will perform an expansion in the parameter R in order to arrive at a system of equations which can be handled analytically. As discussed in the introduction we thereby want to illustrate that the nonlinear system derived in § 3, at least in a linear subrange, defines a well-posed boundary-value problem.

Let us begin with a short discussion of the relation between R and the boundary conditions on ϕ^I derived in § 3. The forcing is provided by the inhomogeneous boundary conditions (3.20) and (3.23). Suppose first that

$$(sL)_v = 0, \quad (sL)_\pm = \text{constant}, \quad T_\pm = \text{constant}. \tag{4.1a}$$

Physically this is the 'trivial' case with perfectly insulated side-wall and constant temperature at the top and bottom. It is intuitively obvious and easily shown from (2.1) that in this case all motions vanish and that T^I is a function of z only, i.e.

$$(u, v, w,) = 0, \quad R = 0, \quad T^I = T^I_{(z)}. \tag{4.1b}$$

The intensity of motion in our system, i.e. the size of R, is associated with deviations from the state defined by (4.1). If $(sL)_v \neq 0$ we obtain directly from (3.20)

$$R \lesssim (sL)_v. \tag{4.2a}$$

The type of forcing defined by (4.2a) corresponds to fluid being pushed out of or sucked into the vertical boundary layer, creating a meridional circulation which through the action of the Coriolis force accelerates a zonal vortex motion.

We note that $(sL)_v$ determines the boundary layer flux while R is related to the variation of this flux. Thus in special cases we may have $R \ll (sL)_v$, which motivates the inequality sign in (4.2a). The forcing on the horizontal boundaries is associated with horizontal variations in $(sL)_\pm$ and/or $T_\pm$.

From (3.3a, c) we have $R \sim T^I_r$. With this relation in mind we find from (3.23) that

$$R \sim \max\left(\frac{\partial}{\partial r}(sL)_\pm, \frac{\partial}{\partial r}T_\pm\right). \tag{4.2b}$$

If the maximum value of R as given by (4.2) is small we may expand our system of equations in R.

We thus introduce

$$T^I = T^{(0)} + RT^{(1)} + \ldots, \tag{4.3a}$$

$$v^I = v^{(1)} + Rv^{(2)} \ldots, \tag{4.3b}$$

$$u^I = u^{(1)} + Ru^{(2)} \ldots, \tag{4.3c}$$

$$w^I = R^{-1}w^{(0)} + w^{(1)} + \ldots. \tag{4.3d}$$

For $(sL)_v$, T_v, $T_\pm$ we choose the following form

$$(sL)_v = (sL)_v^{(0)} + R(sL)_v^{(1)} + \ldots, \tag{4.4a}$$

$$T_v = T_v^{(0)} + RT_v^{(1)} + \ldots, \tag{4.4b}$$

$$T_\pm = T_\pm^{(0)} + RT_\pm^{(1)} + \ldots. \tag{4.4c}$$

For the properties of the horizontal boundaries expressed by $(sL)_\pm$ we introduce two alternatives

$$\text{(i)} \quad (sL)_\pm = (sL)^{(0)}_\pm + R(sL)^{(1)}_\pm + \ldots, \tag{4.4d}$$

$$\text{(ii)} \quad (sL)_\pm \gg R^{-1}. \tag{4.4e}$$

Regarding the form of the expansion (4.4) we note that $T^I \sim 1$ since T^I is scaled with a given external quantity ΔT, $v^I \sim 1$ from the definition of R, $u^I \sim 1$ follows naturally from (3.3*b*) while w^I may be of order 1 or R^{-1} in view of (3.3*d*, *e*).

Introducing (4.3) in (3.3) we obtain after some manipulation

$$T^{(0)} = T^{(0)}(z), \tag{4.5a}$$

$$w^{(0)} = \text{constant}, \tag{4.5b}$$

$$\sigma w^{(0)} T^{(0)}_z = T^{(0)}_{zz}, \tag{4.5c}$$

and

$$v^{(1)}_z = BT^{(1)}_r, \tag{4.6a}$$

$$w^{(0)} v^{(1)}_z + u^{(1)} = \left(\nabla^2 - \frac{1}{r^2}\right) v^{(1)} \tag{4.6b}$$

$$\frac{1}{r}(ru^{(1)})_r + w^{(1)}_z = 0, \tag{4.6c}$$

$$\sigma(w^{(0)} T^{(1)}_z + w^{(1)} T^{(0)}_z) = \nabla^2 T^{(1)}. \tag{4.6d}$$

Introducing (4.3) in the boundary conditions we obtain from (3.20)

$$\frac{d}{dz}\left(\frac{(sL)^{(0)}_v (T^{(0)} - T^{(0)}_v)}{T^0_z}\right) = 0 \quad \text{on} \quad r = 1 \tag{4.7a}$$

from (3.23) with $(sL)_\pm \sim 1$,

$$\text{(i)} \quad T^{(0)}_z = \mp (sL)^{(0)}_\pm (T^{(0)} - T^{(0)}_\pm) \quad \text{at} \quad z = \pm h; \tag{4.7b}$$

and with $(sL)_\pm \gg R^{-1}$

$$\text{(ii)} \quad T^{(0)} = T^{(0)}_\pm \quad \text{at} \quad z = \pm h; \tag{4.7c}$$

from (3.24)

$$\tfrac{1}{2} w^{(0)} - \frac{1}{\sigma}\left(\frac{(sL)^{(0)}_v (T^{(0)} - T^{(0)}_v)}{T^{(0)}_z}\right)_{r=1} = \frac{M_0}{2\pi} \quad \text{when} \quad -h < z < h. \tag{4.7d}$$

Examining the boundary conditions further we obtain the following: from (3.18)

$$v^{(1)} = 0 \quad \text{on} \quad r = 1; \tag{4.8a}$$

from (3.20) to second order

$$u^{(1)} = \frac{-1}{\sigma}\frac{d}{dz}\left((sL)^{(1)}_v \frac{T^{(0)} - T^{(0)}_v}{T^{(0)}_z} + (sL)^{(0)}_v \frac{T^{(1)} - T^{(1)}_v}{T^{(0)}_z} - (sL)^{(0)}_v \frac{T^{(0)} - T^{(0)}_v}{(T^{(0)}_z)^2} T^{(1)}_z\right) \quad \text{on} \quad r = 1; \tag{4.8b}$$

from (3.21)

$$v^{(1)} = 0 \quad \text{on} \quad z = \pm h; \tag{4.8c}$$

from (3.23) with $(sL)_\pm \sim 1$ to second order

$$\text{(i)}\quad T_z^{(1)} + \sigma \frac{r}{2} w^{(0)} T_r^{(1)} = \mp (sL)_\pm^{(1)} (T^{(0)} - T_\pm^{(0)}) \mp (sL)_\pm^{(0)} (T^{(1)} - T_\pm^{(1)}) \quad \text{on} \quad z = \pm h \quad (4.8d)$$

and with $(sL)_\pm \gg R^{-1}$,

$$\text{(ii)}\quad T^{(1)} = T_\pm^{(1)} \quad \text{on} \quad z = \pm h; \qquad (4.8e)$$

from (3.24) to second order

$$\int_0^1 w^{(1)} r\,dr - \frac{1}{\sigma}\left((sL)_v^{(1)} \frac{T^{(0)} - T_v^{(0)}}{T_z^{(0)}} + (sL)_v^{(0)} \frac{T^{(1)} - T_v^{(1)}}{T_z^{(0)}} - (sL)_v^{0} \frac{T^{(0)} - T_v^{(0)}}{(T_z^{(0)})^2} T_z^{(1)} \right) \quad \text{when} \quad -h < z < h. \qquad (4.8f)$$

Equations (4.5) and (4.7) determine the 'basic' field $(T_{(z)}^{(0)}, w^{(0)})$ while the perturbations $(T^{(1)}, v^{(1)}, u^{(1)}, w^{(1)})$ are determined from (4.6) and (4.8).

It is immediately clear from (4.5) that $T_z^{(0)}$ is either a linear or an exponential function of z. This is otherwise stated: If a rotating non-homogeneous fluid is characterized by $(B, \sigma) \sim 1$ then the only allowed strongly stratified states (i.e. with $T_r^I \ll T_z^I$) either have a linear or an exponential density distribution. The conclusion is independent of the boundary conditions and thus valid for more generally shaped regions than the cylinder considered here.

In general (4.7a) requires that

$$(sL)_v^{(0)} \frac{T^{(0)} - T_v^{(0)}}{T_z^{(0)}} = 0,$$

i.e. that $(sL)_v^{(0)} = 0$. Under special circumstances (4.7a) may be satisfied even though $(sL)_v^{(0)} \neq 0$. In case $(sL)_v^{(0)} = 0$ we find from (4.7d) that $M_0 = 0$ implies $w^{(0)} = 0$. This suggests separate study of three cases:

$$\begin{array}{llll} \text{(A)} & (sL)_v^{(0)} = 0, & M_0 = 0 & (w^{(0)} = 0); \\ \text{(B)} & (sL)_v^{(0)} = 0, & M_0 \neq 0 & (w^{(0)} \neq 0); \\ \text{(C)} & (sL)_v^{(0)} \neq 0, & M_0 = 0 & (w^{(0)} \neq 0). \end{array}$$

Case A: *Linear basic stratification*

In general (4.7a) implies that

$$(sL)_v^{(0)} = 0. \qquad (4.9a)$$

The special case when (4.7a) does not have this consequence will be discussed separately in case C below. We also assume

$$M_0 = 0. \qquad (4.9b)$$

From (4.5) and (4.7) we then find

$$w^{(0)} = 0, \quad T^{(0)} = C + Dz, \qquad (4.10a, b)$$

where C and D are determined either through (4.7b) or (4.7c). We conclude that we must require

$$\frac{d}{dr}(T_\pm^{(0)}, sL_\pm^{(0)}) = 0. \qquad (4.10c)$$

Eliminating $T^{(1)}$ from (4.6a, d) and applying (4.10) we obtain from (4.6b, d)

$$\sigma B w_r^{(1)} T_z^{(0)} = \left(\nabla^2 - \frac{1}{r^2}\right) v_z^{(1)},$$

$$u_z^{(1)} = \left(\nabla^2 - \frac{1}{r^2}\right) v_z^{(1)},$$

or

$$\sigma T_z^{(0)} B w_r^{(1)} - u_z^{(1)} = 0. \tag{4.11}$$

This equation illustrates in a simple way the inter-dependence forced upon $(u^{(1)}, w^{(1)})$ by the thermal-wind relation (4.6a). Combining (4.11) with the continuity equation (4.6e) we obtain

$$\sigma T_z^{(0)} B \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}(ru^{(1)}) + \frac{\partial^2}{\partial z^2} u^{(1)} = 0. \tag{4.12a}$$

Furthermore from (4.6b) we have

$$\left(\nabla^2 - \frac{1}{r^2}\right) v^{(1)} = u^{(1)}. \tag{4.12b}$$

Equation (4.12) may of course be combined into a single equation for $v^{(1)}$.

Let us now derive boundary conditions in a suitable form for equations (4.12). Making use of (4.9) and (4.10) we obtain from (4.8a, b)

$$v^{(1)} = 0 \quad \text{on} \quad r = 1, \tag{4.13a}$$

$$u^{(1)} = -\frac{1}{\sigma}\frac{d}{dz}\left((sL)_v^{(1)} \frac{T^{(0)} - T_v^{(0)}}{T_z^{(0)}}\right) \quad \text{on} \quad r = 1, \tag{4.13b}$$

where we note that the right-hand side of (4.13b) is a known function of z. Also making use of the thermal-wind relation we obtain from (4.8c, d, e)

$$v^{(1)} = 0, \tag{4.13c}$$

$$\text{(i)} \quad (sL)_\pm^{(0)} v_z^{(1)} \pm v_{zz}^{(1)} = -B\left((T^{(0)} - T_\pm^{(0)}) \frac{d}{dr}(sL)_\pm^{(1)} - (sL)_\pm^{(0)} \frac{d}{dr} T_\pm^{(1)}\right) \tag{4.13d}$$

$$\text{(ii)} \quad v_z^{(1)} = B \frac{d}{dr} T_\pm^{(1)}, \tag{4.13e}$$

where we note that (4.13d) should be used when $(sL)_\pm \sim 1$ and (4.13e) when $(sL)_\pm \gg R^{-1}$. Finally we obtain from (4.8f)

$$\sigma \int_0^1 w^{(1)} r \, dr = (sL)_v^{(1)} \frac{T^{(0)} - T_v^{(0)}}{T_z^{(0)}}, \quad -h < z < h,$$

or when combined with (4.6d)

$$\sigma \int_0^1 r \nabla^2 T^{(1)} dr = (sL)_v^{(1)} (T^{(0)} - T_v^{(0)}). \tag{4.14}$$

We will comment below on this last condition. Equations (4.12) together with conditions (4.13) form a perfectly well-behaved boundary-value problem which completely determines $u^{(1)}$ and $v^{(1)}$. If however we then want to solve for $w^{(1)}$ and $T^{(1)}$ through the use of (4.6a) and (4.6c) we will find that the solution for $T^{(1)}$ will contain an arbitrary function of z while $w^{(1)}$ will contain an arbitrary function of r. The extra condition (4.14) serves to remove this indeterminacy.

Case B: *Exponential basic stratification with non-zero net flow* M_0

Again we 'let' (4.7a) imply

$$(sL)_v^{(0)} = 0 \tag{4.15a}$$

but we now allow

$$M_0 \neq 0. \tag{4.15b}$$

From (4.5) and (4.7) we obtain

$$w^{(0)} = M_0/2\pi, \tag{4.16a}$$

$$T^{(0)} = F\exp\frac{\sigma M_0}{2\pi}z + G, \tag{4.16b}$$

where F and G are determined either through (4.7b) or (4.7c), subject to the restriction (4.10c).

In this case there is no equivalence to equations (4.11) and (4.12a) because of the factor σ in (4.6d). We may however derive a single equation for $v^{(1)}$. We thus have

$$B\sigma\frac{\partial}{\partial r}r\frac{\partial}{\partial r}r\left(\nabla^2 - \frac{1}{r^2} - w^{(0)}\frac{\partial}{\partial z}\right)v^{(1)} + \frac{\partial}{\partial z}\frac{1}{T_z^{(0)}}\frac{\partial}{\partial z}\left(\nabla^2 - \frac{1}{r^2} - \sigma w^{(0)}\frac{\partial}{\partial z}\right)v^{(1)} = 0. \tag{4.17}$$

We thus want boundary conditions in terms of $v^{(1)}$ if possible. Such boundary conditions may in fact be derived in a straightforward manner. They become fairly complicated and will not be written down here. We note only that the second term in (4.8d) which reflects the heat advection of the Ekman layers is of importance in the case with $(sL)_\pm \sim 1$ i.e. with semiconducting horizontal boundaries.

Case C: *Exponential stratification with non-divergent vertical boundary layer*

Let us explore the possibility that (4.7a) is in fact satisfied even though $(sL)_v^{(0)} \neq 0$. We will thus consider the case

$$(sL)_v^0 \neq 0, \quad M_0 = 0. \tag{4.18a, b}$$

We are not interested in the 'trivial' case $T^{(0)} = T_v^{(0)}$. Thus we require that

$$(sL)_v^0(T^{(0)} - T_v^{(0)})\,(T_z^{(0)})^{-1}$$

be a constant or in view of (4.7d)

$$w^{(0)} = \frac{2}{\sigma}\,(sL)_v^{(0)}\left(\frac{T^{(0)} - T_v^{(0)}}{T_z^{(0)}}\right)_{r=1} = \text{constant}. \tag{4.19}$$

From (4.5) we obtain, as in the foregoing case B, that

$$T^{(0)}(z) = F\exp\sigma w^{(0)}z + G \tag{4.20a}$$

(4.19) and (4.20a) are compatible if

$$\sigma w^{(0)} = \pm\,(2(sL)_v^{(0)})^{\frac{1}{2}}, \quad T_v^{(0)} = G, \tag{4.20b}$$

which obviously implies that $T_v^{(0)}$ as well as $(sL)_v^{(0)}$ are constants. We obtain

$$T^{(0)}_{(z)} = T_v^{(0)} + F \exp \pm \gamma z, \tag{4.21a}$$

where

$$\gamma = (2(sL)_v^{(0)})^{\frac{1}{2}}, \tag{4.21b}$$

where we have to keep in mind that we can accept only gravitationally stable solutions. This implies that the plus sign should be used when $F > 0$ and *vice versa.*

The problem is now that we must satisfy two boundary conditions as given by (4.7 *b*) or (4.7 *c*). Having only one free constant in (4.21) we thus have an over-determined system. Consequently (4.20) can be accepted as a solution if and only if the boundary conditions are chosen appropriately. From (4.21) and (4.7 *b*) or (4.7 *c*) we may derive the condition to be satisfied by the parameters in the boundary conditions. We obtain

$$\frac{T_+^{(0)} - T_v^{(0)}}{T_-^0 - T_v^0} = \alpha \exp \gamma 2h, \tag{4.22a}$$

where

$$\alpha = \left(1 + \frac{\gamma}{(sL)_+^{(0)}}\right)\left(1 - \frac{\gamma}{(sL)_-^{(0)}}\right)^{-1} \quad \text{when} \quad (sL)_\pm \sim 1 \tag{4.22b}$$

and

$$\alpha = 1 \quad \text{when} \quad (sL)_\pm \gg R^{-1}. \tag{4.22c}$$

The system described by (4.18)–(4.22) is not a very general one. In fact we have been forced to assume:

(i) The insulation of the side walls as well as the temperature outside the side wall (i.e. the parameters $(sL)_v^{(0)}$, $T_v^{(0)}$) are independent of z.

(ii) Likewise the top and bottom parameters $(sL)_\pm^{(0)}$, $T_\pm^0$ must be constant.

(iii) Finally these thermal parameters have to satisfy the compatibility condition given by (4.22).

The situation may be understood in the following way:

The interior heat balance and the requirement of a non-divergent buoyancy layer locks the interior temperature field into an exponential variation, always decaying towards the side-wall temperature $T_v^{(0)}$ with a fixed rate of decay. The amplitude of the exponential is however free. We may thus adjust the exponential to the temperature prescribed at one of the boundaries $z = \pm h$. The compatibility condition (4.22) then guarantees that the exponential will automatically achieve the right temperature at the opposite boundary.

We can also make the following interpretation. If all conditions imposed in this section except the compatibility condition (4.22) are satisfied our solution will not be valid. In fact the linearization breaks down and we expect strong vortex motion associated with fully-developed radial-temperature variations. We may thus make the qualitative prediction that by simply changing e.g. one of the temperatures $T_\pm^{(0)}$, $T_v^{(0)}$ until (4.22) is satisfied, we will be able to slow down this vortex motion by an order of magnitude and bring the system into a state with small horizontal temperature gradients.

Verification of this prediction should make a simple and appealing experiment.

Turning now to the perturbation fields $(v^{(1)}, T^{(1)})$ we first note that the 'thermal'

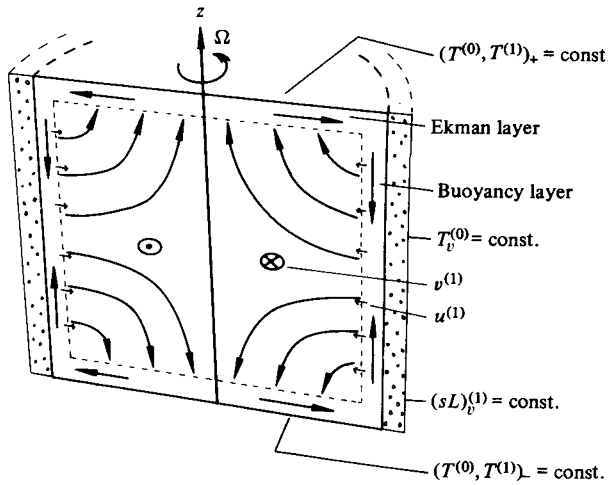

FIGURE 2. General illustration of the simple special case discussed in § 5. The system is stratified by a vertical temperature contrast. The side boundary has a small but finite conductance and is kept at constant temperature on its outside. The result is a buoyancy layer with height independent divergence forcing a vortex in the interior as indicated in the figure.

boundary condition (4.8*b*) cannot be expressed in terms of $v^{(1)}$ in this case. We may however easily derive an elliptic equation (similar to (4.17)) and associated boundary conditions in terms of $T^{(1)}$, which form a well-posed boundary-value problem.

5. A simple but basic example

Let us now apply our theory to a particularly straightforward case, i.e. a right circular cylinder without mean flow, with a warm upper and a cold lower boundary and semiconducting sidewall facing on the outside a constant ambient temperature (see figure 2). We also assume that the side wall has a small enough conductivity for the system to belong to the category treated under *A* in § 4 (figure 2). We thus have

$$M_0 = 0, \tag{5.1a}$$

$$(sL)_v^{(0)} = 0, \tag{5.1b}$$

$$T_v^{(0)} = \text{constant}, \tag{5.1c}$$

$$(sL)_v^{(1)} = c_0\,(=\text{constant}), \tag{5.1d}$$

$$(sL)_\pm \gg R^{-1}, \tag{5.1e}$$

$$(T_\pm^{(0)}, T_\pm^{(1)}) = \text{constant}. \tag{5.1f}$$

From (4.10) we find

$$T^{(0)} = T_-^{(0)} + z \tag{5.2}$$

having assumed (without loss of generality)

$$T_+^{(0)} - T_-^{(0)} = 1.$$

From (5.1) and (4.13) we obtain the following boundary conditions to be applied on equation (4.12).

$$v^{(1)} = 0 \quad \text{on} \quad r = 1, \tag{5.3a}$$

$$u^{(1)} = -\frac{1}{\sigma} c_0 = -c \quad \text{on} \quad r = 1, \tag{5.3b}$$

$$v^{(1)} = 0 \quad \text{on} \quad z = \pm h, \tag{5.3c}$$

$$v_z^{(1)} = 0 \quad \text{on} \quad z = \pm h. \tag{5.3d}$$

Furthermore we note that the continuity equation (4.6c) implies that nonsingular solutions satisfy

$$u^{(1)} = 0 \quad \text{at} \quad r = 0. \tag{5.3e}$$

A solution to (4.12a), which satisfies the inhomogeneous boundary condition (5.3b) and the regularity condition (5.3e) but is otherwise completely general may be written:

$$u^{(1)} = -cr + \sum_n (D_n^* \sinh \alpha_n . \beta z + E_n^* \cosh \alpha_n \beta z) J_1(\alpha_n . r) \tag{5.4a}$$

where

$$\beta^2 = B\sigma \tag{5.4b}$$

and where α_n is determined by the condition

$$J_1(\alpha_n) = 0; \tag{5.4c}$$

D_n^* and E_n^* are arbitrary. Introducing (5.4) into (4.12b) and making use of the homogeneous boundary condition (5.3a) we find that for $v^{(1)}$

$$v^{(1)} = \frac{cr}{8}(1-r^2) + \Sigma(D_n \sinh \alpha_n \beta z + E_n \cosh \alpha_n \beta z) J_1(\alpha_n r)$$
$$+ \Sigma(F_n \sinh \alpha_n z + G_n \cosh \alpha_n z) J_1(\alpha_n r), \tag{5.5a}$$

where

$$(D_n, E_n) = \frac{1}{(\beta^2-1)\alpha_n^2}(D_n^*, E_n^*). \tag{5.5b}$$

The unknown constants (D_n, E_n, F_n, G_n) in (5.5) should now be determined from the remaining boundary conditions (5.3c, d) on the horizontal boundaries. This may readily be done in terms of the coefficients A_n defined by

$$\frac{c}{8} r(1-r^2) = \Sigma A_n J_1(\alpha_n r). \tag{5.6}$$

Introducing (5.5) and (5.6) into (5.3c, d) we obtain

$$\pm aD_n + bE_n \pm cF_n + dG_n = -A_n, \tag{5.7a}$$

$$\beta b D_n \pm \beta a E_n + dF_n \pm cG_n = 0, \tag{5.7b}$$

where

$$(a, b, c, d) = (\sinh \alpha_n \beta h, \cosh \alpha_n \beta h, \sinh \alpha_n h, \cosh \alpha_n h). \tag{5.7c}$$

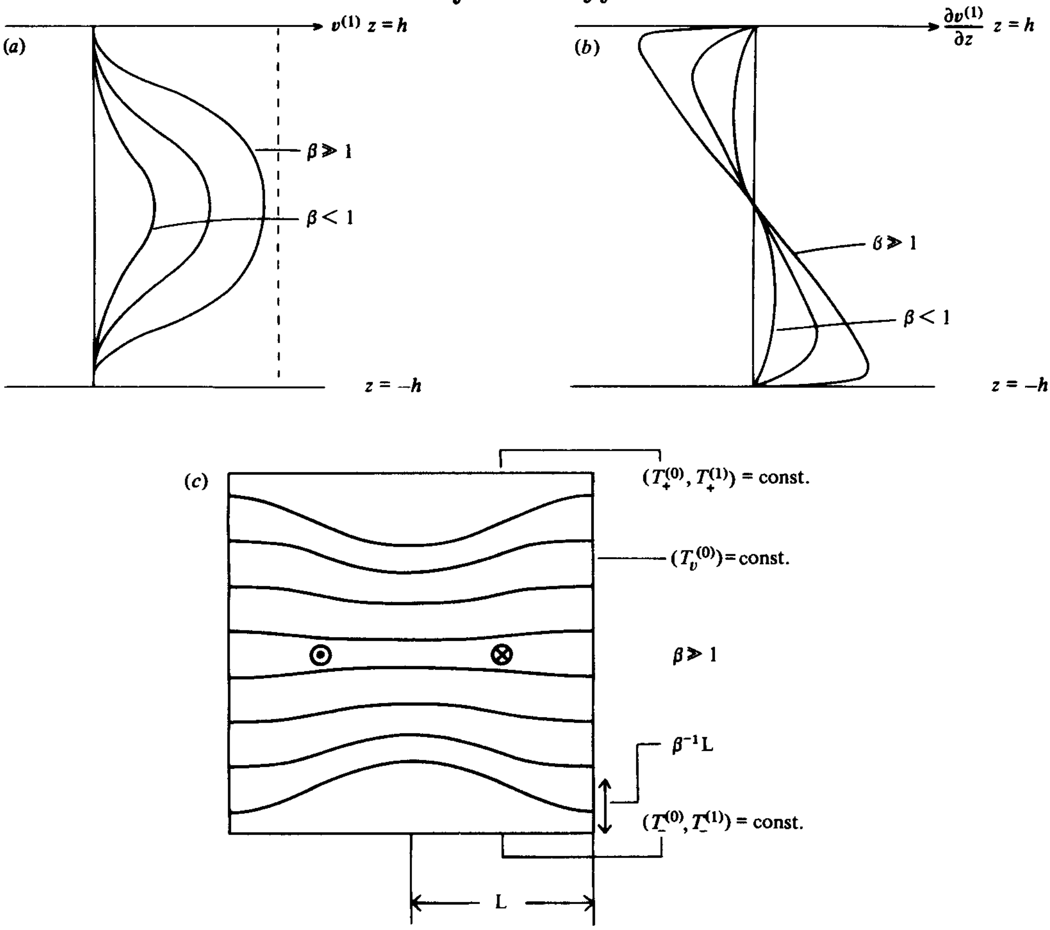

FIGURE 3. Illustration of velocity and temperature fields for the case discussed in §5 and illustrated in figure 2. (a) The zonal velocity profile $(v_{(z)}^{(1)})$ for different values of the parameter $\beta = (B\sigma)^{\frac{1}{2}}$. (b) Illustration of the zonal shear $(\partial v^{(1)}/\partial z)$ for different values of β. Note that the adjustment to zero shear at $z = \pm h$ takes place in narrow regions when $\beta \gg 1$ (the Lineykin depth). (c) The corresponding distortion of the isotherms $T^{(0)} + T^{(1)} =$ const. for a case with $\beta \gg 1$. The characteristic height values are shown in the figure.

Solving for (D_n, E_n, F_n, G_n) we obtain

$$D_n = F_n = 0,$$
$$E_n = A_n \frac{c}{\beta ad - bc},$$
$$G_n = A_n \frac{-\beta a}{\beta ad - bc},$$

we note that the disappearing of D_n and F_n is a result of the vertical symmetry of the boundary conditions on $(u^{(1)}, v^{(1)})$.

6. Discussion

Some qualitative features of the solution described by (5.5)–(5.8) are discussed in this section and illustrated in figures 2 and 3. The vortex is unidirectional with the same sign as the basic rotation. The angular velocity decays smoothly towards the horizontal as well as the vertical boundaries, as illustrated in figure 3(a). The solution has two natural (non-dimensional) length scales (i) the width of the region (~ 1) (ii) the Lineykin depth ($\sim \beta^{-1}$) (Lineykin 1955).

Generally we expect the parameter $\beta = (B\sigma)^{\frac{1}{2}}$ to act as a natural scale ratio in the sense

$$\frac{\partial}{\partial z}\phi^I \sim \beta \frac{\partial}{\partial r}\phi^I.$$

The situation is however not quite that simple because of the presence of two natural scales in the solution. These different length scales show up particularly well in the vertical distribution of the shear $v_z^{(1)}$ when $\beta \gg 1$. We thus find that while $v^{(1)}$ itself adjusts smoothly to the boundary conditions, the shear varies strongly in thin layers of thickness β^{-1} close to $z = \pm h$ as illustrated in figure 3(c). Since the perturbation on the temperature field is directly related to $v_v^{(1)}$ [see equation (4.6a)] we expect that the distortion of the isotherms will have maxima close to $z = \pm h$ when $\beta \gg 1$ (see figure 3c).

We note the difference to the timedependent case (Walin 1969) in which the ratio of scales, in that case given by $B^{\frac{1}{2}}$ instead of $(B\sigma)^{\frac{1}{2}}$, dominates the system completely.

The specific example analysed in § 5 illustrates

(i) some basic features of inhomogeneous rotating fluids subject to stationary thermal forcing,

(ii) that the more general system discussed in §§ 3 and 4, at least in a simple special case, leads to a well-behaved boundary value problem.

We may thus hope that the system derived in § 3 may lend itself to meaningful numerical computations in a flow régime which is perhaps of more than academic interest.

7. Proposed experiments

The analysis in this paper primarily suggests two experiments:

(i) The system analysed in § 5 appears as a particularly simple case for laboratory verification. From experiments one could also judge to what extent the qualitative features of the linear solution of § 5 are changed when the forcing is increased and the interior dynamics becomes nonlinear.

(ii) The special case C of § 4 suggests a particularly simple experiment.

One of the authors (Rahm 1976) has carried out some experiments to which the analysis in this paper applies. In this case the cylinder had a 'perfectly insulated' vertical boundary. The top and bottom plates were semiconducting and held at constant but different temperatures on the outside. The forcing was provided by a variation in the thickness of the bottom plate. The theoretical solution in this case shows a vortex with maximum strength at a height $\sim \beta^{-1}$.

The vortex strength thus grows upwards throughout the Lineykin layer. At greater

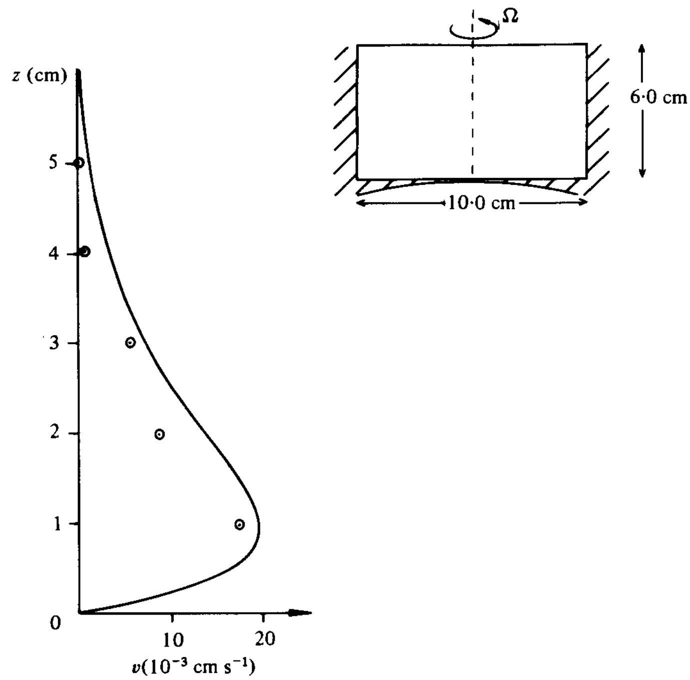

FIGURE 4. The predicted (solid line) and observed (⊙) velocity *vs.* depth (at radius 3·0 cm) for a case where a baroclinic vortex was forced by a non-uniform heat flux through the bottom as described in §7. The basic stratification was linear as the vertical boundary was insulated. The level of maximum velocity is shifted downwards as β is increased. In this case we had $\beta \sim 2$. The dimensions of the experimental apparatus is shown in the upper right corner.

height, in the case $\beta > 1$, the vortex decays with the larger characteristic scale ~ 1, i.e. of the order of the horizontal scale of the region.

The experimental results (see figure 4), although not very accurate, substantiate the theoretical findings and we do feel the need for a more precise laboratory test of our analysis. We also think that the choice of boundary conditions in that experiment is more complicated to realize experimentally than the system analysed in §5.

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